Faculty Working Papers

ON THE USE OF COBB-DOUGLAS SPLINES

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1. Introduction

Since the legendary work of Cobb and Douglas [1], Cobb-Douglas production functions and (to a somewhat lesser degree) Cobb-Douglas utility functions have been popular tools of economists. This popularity can be attributed both to the simplicity and to the wide-applicability of these functions. However, these functions are of course subject to rather severe restrictions. For example, in the production function context returns to scale are non-varying, hence, U-shaped average cost curves are ruled out. Also, homotheticity and unitary elasticities of substitution are required.

This study develops the idea of continuous piecewise Cobb-Douglas functions along the spline function lines discussed in Poirier [10] - [12]. This development will permit U-shaped average cost curves and "piecewise-homotheticity" at the expense of differentiability of the functions along lines parallel to the input axes. However, the unitary elasticities of substitution requirements will remain.

Implicit in this discussion is the belief expressed in Poirier [10] - [12] that for a wide range of problems in economics, the added generality of more complicated functional forms is often best achieved by using continuous piecewise functions. Briefly, the rationale is two-fold. First,

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[1]The author is an Assistant Professor of Economics at the University of Illinois at Urbana-Champaign. The contents of this study rely heavily on Poirier [10, Chapter 4]. Thanks are due Joseph Hotz, Steven Garber, William Greene, and Diane Christensen of the University of Wisconsin at Madison for their data handling and programming assistance which contributed greatly to section 5. Of course any errors are the sole responsibility of the author.

since within each "piece" such functions have simple and familiar forms, the analysis proceeds quite straightforwardly. Indeed all economists are familiar with Cobb-Douglas functions and their properties, and so this previous knowledge can be easily applied in analyzing piecewise Cobb-Douglas functions. Second, changes in the behavior of such functions as one passes from one piece to another are often of primary concern in economic analysis (e.g., the changes in expansion paths discussed in Section 3). Indeed the testability for the existence of such "structural changes" will be an important consideration throughout this discussion.

Organizationally, we will proceed as follows. Section 2 defines a Cobb-Douglas spline. For the sake of simplicity, but not at the expense of generality, a production function context with two inputs, labor and capital, will be used. Section 3 discusses the properties of Cobb-Doublas spline production functions, leaving to section 4 a discussion of the properties of Cobb-Douglas spline utility functions. Finally, section 5 contains an empirical application of a CDS production function.

2. Definition of a Cobb-Douglas Spline

Let the sets $\Delta_L = \{\bar{L}_1 < \bar{L}_2 < \ldots < \bar{L}_{I-1}\}$ and $\Delta_K = \{\bar{K}_1 < \bar{K}_2 < \ldots < \bar{K}_{J-1}\}$ be meshes defining intervals in the labor (L) and capital (K) dimensions. The elements in Δ_L and Δ_K are called knots and they define a rectangular grid in the positive quadrant that consists of IJ rectangles (see Figure 1). A Cobb-Douglas spline (CDS) is a function Q(L,K) which can be defined as

$$(1) \qquad Q(L,K) = \theta_{ij} L^{\alpha_i} K^{\delta_j},$$

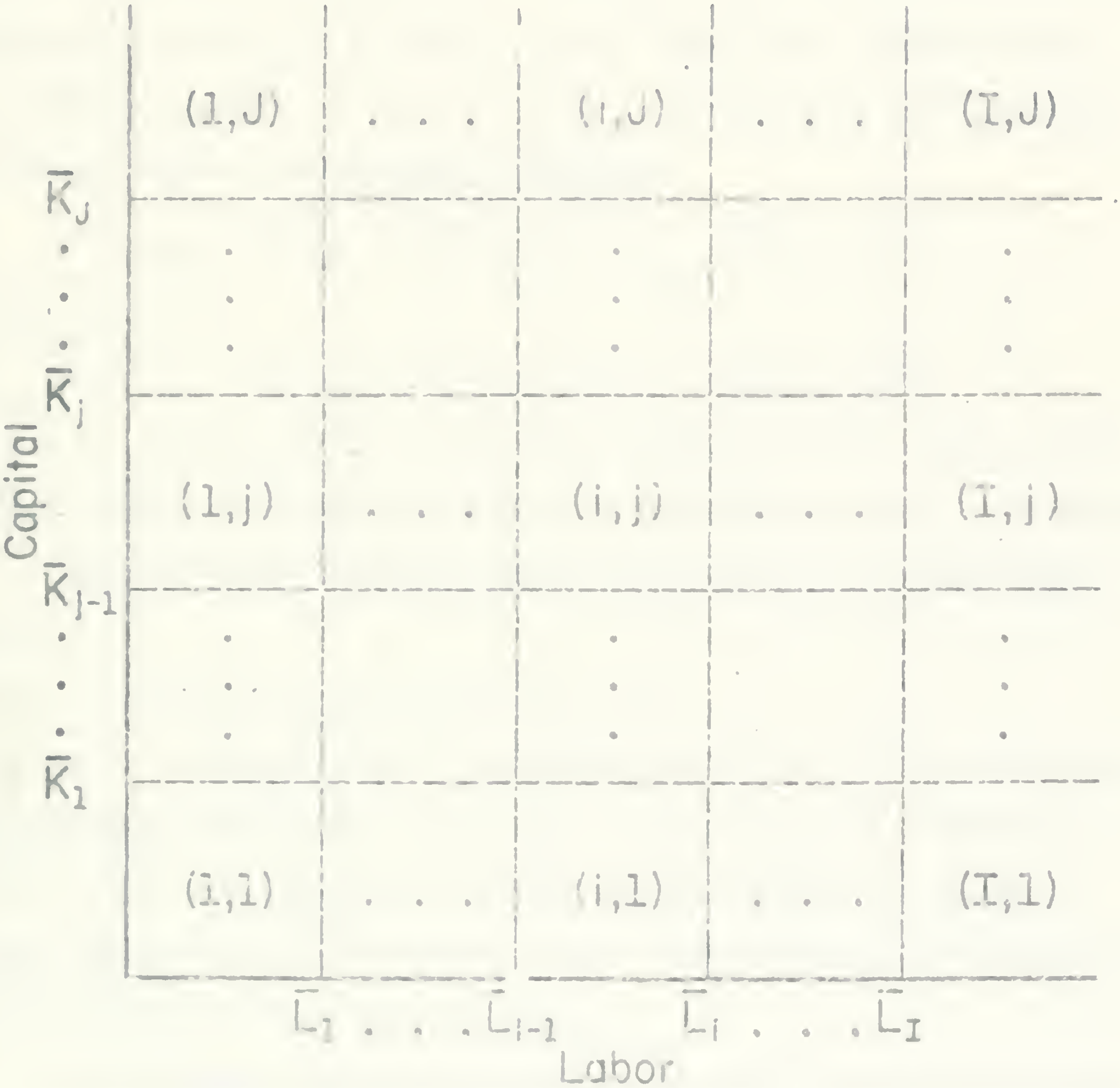

Figure 1: Labor-Capital Grid

where L and K are in rectangle (i,j), the θ_{ij}'s, α_i's, and δ_j's are positive constants, and the θ_{ij}'s are chosen so as to make $Q(L,K)$ continuous over the positive quadrant. Given θ_{11}, the α_i's, and the δ_j's, this continuity requirement implies the <u>continuity conditions</u>

(2) $$\ln \theta_{(i+1)j} = \ln \theta_{ij} + (\alpha_i - \alpha_{i+1}) \ln \bar{L}_i \qquad i=1,2,\ldots,I-1$$

for all j, and

(3) $$\ln \theta_{i(j+1)} = \ln \theta_{ij} + (\delta_j - \delta_{j+1}) \ln \bar{K}_j \qquad j=1,2,\ldots,J-1$$

for all i. In this context the output elasticities of labor (the α_i's) and capital (the δ_j's) are step functions over the meshes Δ_L and Δ_K, respectively.

Often it is more convenient to work with a CDS in its logarithmic form

(4) $$\ln Q = \ln \theta_{ij} + \alpha_i \ln L + \delta_j \ln K$$

for L and K in rectangle (i,j). In formulation (4) ln Q is the sum of two <u>linear splines</u> (see Poirier [10, Chapter 2]), one in the ln L dimension, and one in the ln K dimension. In terms of Poirier [12], (4) is also a <u>bilinear spline</u> with no interaction terms.[2] Geometrically, (4) defines a piecewise-planar or "roof-like" surface.

Alternatively, continuity conditions (2) and (3) can be incorporated into representation (4) in the following manner. Define the variables

$$\tilde{L}_i = \max\,[(\ln L - \ln \bar{L}_i),\ 0] \quad i = 1, 2, \ldots, I-1$$

$$\tilde{K}_j = \max\,[(\ln K - \ln \bar{K}_j),\ 0] \quad j = 1, 2, \ldots, J-1$$

Then for all L and K,

(5) $$\ln Q = \ln \theta_{11} + \alpha_1 \ln L + \delta_1 \ln K + \sum_{i=2}^{I-1} \tilde{\alpha}_i \tilde{L}_i + \sum_{j=2}^{J-1} \tilde{\delta}_j \tilde{K}_j,$$

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[2] For similar comparisons in terms of <u>grafted polynomials</u>, see Fuller [illegible] and Gallant and Fuller [3].

where the parameters $\tilde{\alpha}_i$ $(i=2,3,\ldots,I-1)$ and $\tilde{\delta}_j$ $(j=2,3,\ldots,J-1)$ represent changes in the output elasticities of labor and capital, respectively. Representation (5) is useful in estimation problems in which these changes are of particular interest. For more details, see Poirier [12].

In the following sections, whenever (1) is used in a production theory context, it will be called a CDS production function, and when it is used in a utility theory context, it will be called a CDS utility function.

3. CDS Production Functions

Of foremost importance in discussing the theoretical properties of the CDS production function are its isoquants. For a fixed output level Q_0, the isoquant over rectangle (i,j) is

$$K = \left[\frac{Q_0 \, L^{-\alpha_i}}{\theta_{ij}}\right]^{1/\delta_j} . \tag{6}$$

Throughout the labor-capital dimensions the isoquants are continuous, however, they have "corners" along the grid lines. This first result follows from the continuity of the production function, and the latter is a result of the non-differentiability of the production function along the grid lines. The isoquants are strictly convex iff

$$\alpha_i \geq \alpha_{i+1} \qquad i = 1,2,\ldots,I-1$$

and

$$\delta_j \geq \delta_{j+1} \qquad j = 1,2,\ldots,J-1$$

In other words the isoquants are strictly convex iff each output elasticity is a decreasing step function of its respective input.

In terms of the returns to scale $(\alpha_i + \delta_j)$ of rectangle (i,j), the convexity conditions imply that

$$\alpha_i + \delta_j \le \alpha_m + \delta_n \qquad m=1,2,\ldots,i; \quad n=1,2,\ldots,j$$

and

$$\alpha_i + \delta_j \ge \alpha_m + \delta_n \qquad m=i,i+1,\ldots,I; \quad n=j,j+1,\ldots,J$$

As shown in Figure 2, this means that $Q(L,K)$ exhibits greater returns to scale over all rectangles below and to the left of rectangle (i,j), and smaller returns to scale over all rectangles above and to the right of rectangle (i,j). Comparisons to rectangles above and to the left of rectangle (i,j), and below and to the right of rectangle (i,j) are inconclusive.

It is well known that Cobb-Douglas isoquants give rise to a straight line expansion path since the production function is homothetic. The Cobb-Douglas spline production function has straight line expansion path segments over each individual rectangle, however, these paths exhibit unique behavior along the grid lines.

To see this suppose the convexity condition hold and consider the slope of isoquant (6) in rectangle (i,j), but not along its border:

$$\frac{dK}{dL} = -\frac{\alpha_i}{\delta_j}\frac{K}{L} .$$

The marginal rate of technical substitution (MRTS), $|dK/dL|$, is not well-defined along the borders because the marginal products are not well-defined there as a result of the jump discontinuities in the output elasticities.

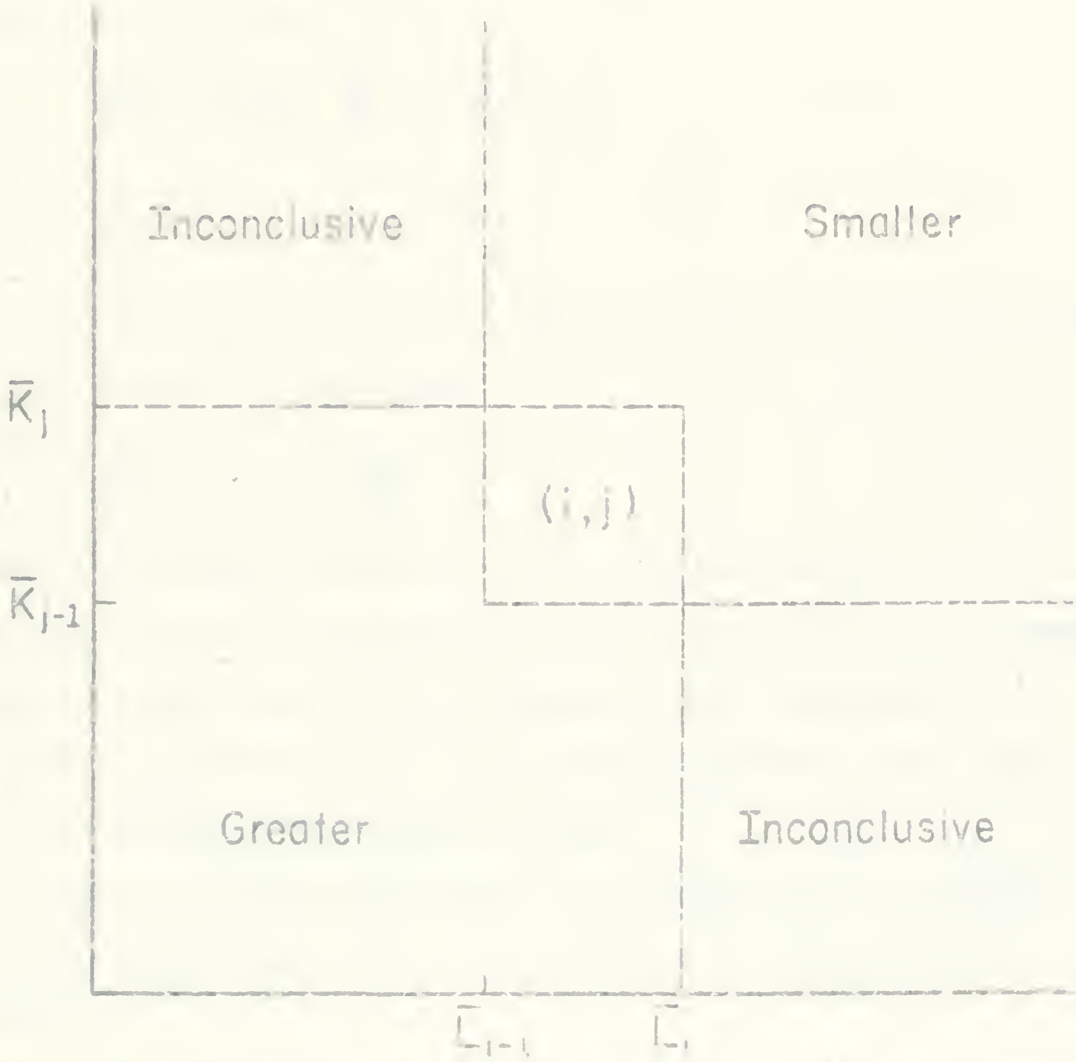

Figure 2: Returns to Scale Comparisons

Considering rectangles (i,j) and $(i+1,j)$, the one-sided derivatives of an isoquant along $L = \bar{L}_i$ are

$$\frac{dK}{dL^+} = \lim_{L \to \bar{L}_i^+} \frac{dK}{dL} = -\frac{\alpha_{i+1}}{\delta_j}\,\frac{K}{\bar{L}_i}$$

$$\frac{dK}{dL^-} = \lim_{L \to \bar{L}_i^-} \frac{dK}{dL} = -\frac{\alpha_i}{\delta_j}\,\frac{K}{\bar{L}_i}\ .$$

Letting w denote the price of labor and r the price of capital, then for all price ratios w/r such that

$$(7) \qquad \left|\frac{dK}{dL^+}\right| < \left|\frac{w}{r}\right| < \left|\frac{dK}{dL^-}\right|,$$

the first order conditions for the firm's output maximization or cost minimization problem are not satisfied, i.e., the marginal rate of technical substitution does not equal the price ratio. Hence, compensated price changes within the bounds of (7) do not change the optimal input combination for producing a fixed level of output.

Fixing w/r and expanding output, any expansion path in rectangle (i,j) approaching the grid line $L = \bar{L}_i$, does so at a capital level K^* satisfying

$$\left|\frac{dK}{dL^-}\right| = \frac{\alpha_i}{\delta_j}\,\frac{K^*}{\bar{L}_i} = \frac{w}{r}\ ,$$

or solving for K^*,

$$(8) \qquad K^* = \frac{\delta_j \bar{L}_i}{\alpha_i}\left(\frac{w}{r}\right)$$

Since the output elasticity of labor over rectangle $(i+1,j)$ is less than it is over rectangle (i,j), increases in output beyond $Q(\bar{L}_i, K^*)$ are met by increasing only the capital input until either the righthand MRTS equals the input price ratio, or until the output elasticity of capital

drops, i.e., until the grid line $K = \bar{K}_j$ is reached. If there does exist $K^{**} < \bar{K}_j$ such that

$$\left|\frac{dK}{dL}\right| = \frac{\alpha_{i+1}}{\delta_j} \frac{K^{**}}{\bar{L}_i} = \frac{w}{r} ,$$

or solving for K^{**},

$$K^{**} = \frac{\delta_j \bar{L}_i}{\alpha_{i+1}} \frac{w}{r} ,$$

then the expansion path in rectangle $(i+1,j)$ for outputs $Q(L,K) > Q(\bar{L}_i, K^{**})$ is a straight line with a slope $(\delta_j w/\alpha_{i+1} r)$ greater than the slope $(\delta_j w/\alpha_i r)$ of the expansion path in rectangle (i,j). This is illustrated in Figure 3(a.) for the simple case $I = J = 2$, $i = j = 1$.

However, if the grid line $K = \bar{K}_j$ is reached first, then the expansion path for outputs $Q(L,K) > Q(\bar{L}_i, \bar{K}_j)$ continues along $L = \bar{L}_i$ iff

$$(9) \qquad \lim_{\substack{L \to \bar{L}_i^- \\ K \to \bar{K}_j^+}} \left|\frac{dK}{dL}\right| = \frac{\alpha_{i+1}}{\delta_{j+1}} \frac{\bar{K}_j}{\bar{L}_i} < \frac{w}{r} < \lim_{\substack{L \to \bar{L}_i^- \\ K \to \bar{K}_j^+}} \left|\frac{dK}{dL}\right| = \frac{\alpha_i}{\delta_{j+1}} \frac{\bar{K}_j}{\bar{L}_i} ;$$

and proceeds along $K = \bar{K}_j$ iff

$$(10) \qquad \lim_{\substack{L \to \bar{L}_i^+ \\ K \to \bar{K}_j^-}} \left|\frac{dK}{dL}\right| = \frac{\alpha_{i+1}}{\delta_j} \frac{\bar{K}_j}{\bar{L}_i} < \frac{w}{r} < \lim_{\substack{L \to \bar{L}_i^+ \\ K \to \bar{K}_j^+}} \left|\frac{dK}{dL}\right| = \frac{\alpha_{i+1}}{\delta_{j+1}} \frac{\bar{K}_j}{\bar{L}_i} .$$

If (9) holds, then the expansion path continues along $L = \bar{L}_i$ until either the righthand MRTS equals the input price ratio at a capital level $K^{***} < \bar{K}_{j+1}$, or until the next grid line $K = \bar{K}_{j+1}$ is reached — upon which the above procedure is repeated. The first of these two possibilities is illustrated in Figure 3(b.).

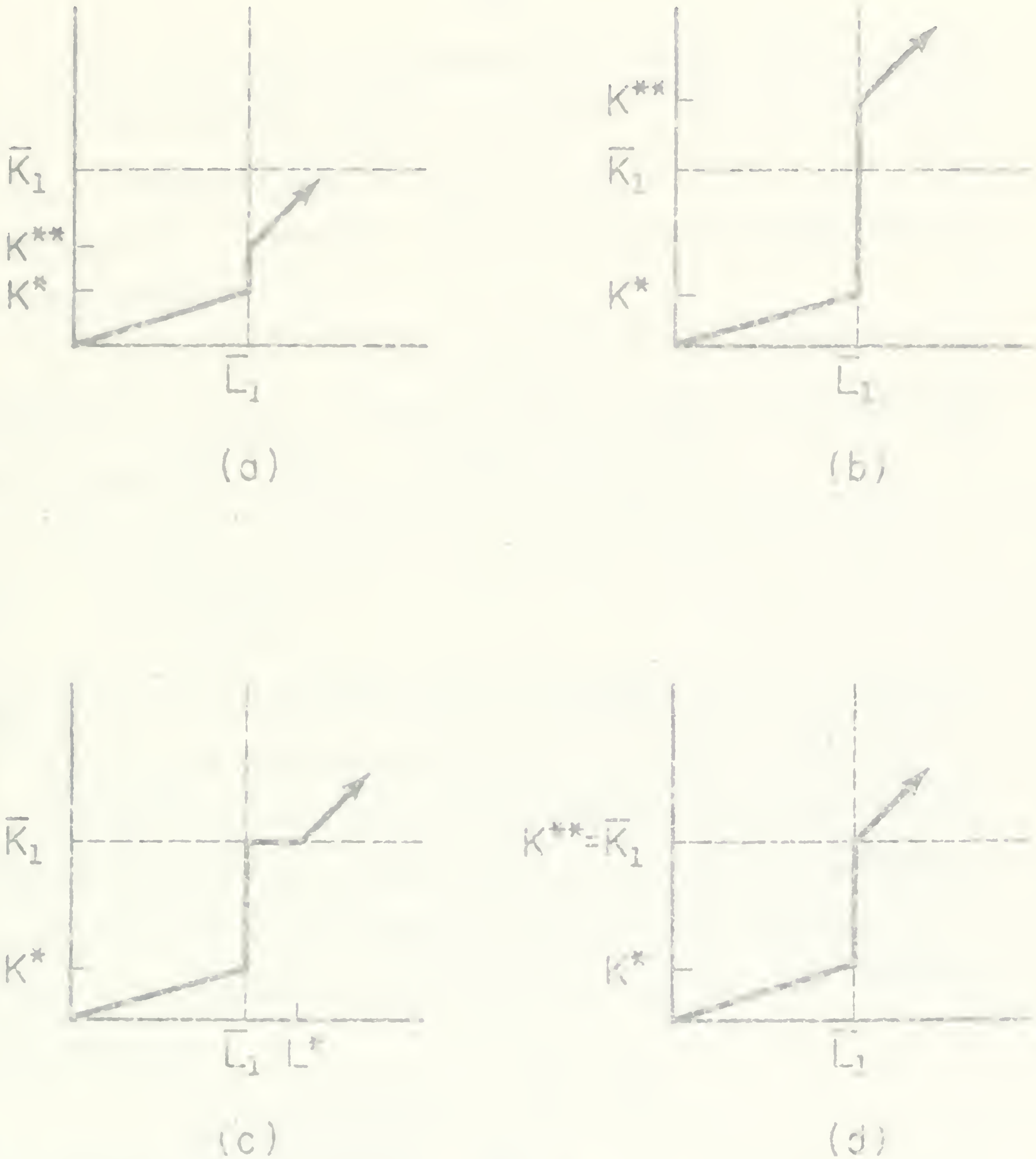

Figure 3: Expansion Paths

Figure 3: Expansion Paths

If (10) holds, then the expansion path continues along $K = \bar{K}_j$ until either the right hand MRTS equals the input price ratio at a labor level $L^* < \bar{L}_{i+1}$, or until the next grid line $L = \bar{L}_{i+1}$ is reached — upon which the above procedure is repeated. The first of these two possibilities is illustrated in Figure 3(c.).

If the grid line $L = \bar{L}_{i+1}$ is reached first, then analogous to the possibilities at $(\bar{L}_i, \bar{K}_j)$, the expansion path for $Q(L,K) \geq Q(\bar{L}_{i+1}, \bar{K}_j)$ may proceed along either $L = \bar{L}_{i+1}$ or continue along $K = \bar{K}_j$.

The remaining possibility not covered by (9) or (10) is that

$$(11) \qquad \frac{w}{r} = \frac{\alpha_{i+1}}{\beta_{j+1}} \frac{\bar{K}_j}{\bar{L}_i}$$

If (11) holds, then the expansion path in rectangle (i,j) is a ray with slope $(\beta_{j+1} w / \alpha_{i+1} r)$ emanating from $(\bar{L}_i, \bar{K}_j)$ as shown in Figure 3(d.).

The cost function associated with a CDS production function is a continuous piecewise function whose pieces are conventional Cobb Douglas cost functions whenever the expansion path lies in the interior of a rectangle. For Q produced in the interior of rectangle (i,j), the CDS cost function is

$$(12) \qquad C = r\left(\frac{\alpha_i + \beta_j}{\beta_j}\right) \left[\left(\frac{\beta_j w}{\alpha_i r}\right)^{\alpha_i} \frac{Q}{A_{ij}}\right]^{1/(\alpha_i + \beta_j)} .$$

If Q is produced along the grid line $L = \bar{L}_i$, then the cost function is

$$(13) \qquad C = w\bar{L}_i + r\left[\frac{Q}{A_{ij}\bar{L}_i^{\alpha_i}}\right]^{1/\beta_j} ,$$

and if it is produced along the grid line $K = \bar{K}_j$, then the cost function is

$$\text{(14)} \qquad C = w \left[\frac{Q}{\theta_{ij} \bar{K}_j^{\beta_j}} \right]^{1/\alpha_i} + r\bar{K}_j .$$

In the case of (13), as $Q \to Q(\bar{L}_i, K^*)$, (12) approaches (13). Similarly, as $Q \to Q(L^*, \bar{K}_j)$, (12) approaches (14). Hence, the cost function is continuous.

The average cost function associated with (12) - (14) is obtained by dividing each equation by $Q (Q \neq 0)$. With regard to (12), the average cost function is increasing, constant, or decreasing, depending on whether Q is produced in a rectangle with decreasing, constant, or increasing returns to scale. Since the cost function is continuous, so is the average cost function.

The marginal cost function associated with (12) - (14) is obtained by differentiating each equation with respect to Q. At output levels corresponding to "kinks" in the expansion path marginal cost is not defined since the left and right hand derivatives are not in general equal. Of course with regard to (12), the marginal cost segment is of the conventional type for a Cobb-Douglas production function.

4. CDS Utility Functions

Not surprisingly, the CDS has interesting applications in utility theory as well as in production theory. For example consider the CDS utility function

$$U(Y, L) = \begin{cases} \theta_1 Y^{\alpha_1} L^{\beta}, & \text{if } Y \leq \bar{Y} \\ \theta_2 Y^{\alpha_2} L^{\beta}, & \text{if } Y > \bar{Y} \end{cases}$$

where Y is an individual's earned weekly income, L is leisure time, $\bar{Y}$ = $80/week is an income level at which income may undergo a drop in its utility elasticity (i.e., $\alpha_1 \geq \alpha_2$), and is $\beta_0 + \ln \beta_1 + (\alpha_1 - \alpha_2)\bar{Y}$. Following an example by Rees [11, pp. 158-160], figure 4 shows the labor-leisure choices of a working male who has a straight-time wage rate of $2.00 per hour, and who is free to vary his hours worked.[3] In the initial situation the worker is in equilibrium at point C on indifference curve I_0 corresponding to 40 hours of work a week and an income equalling the critical value of $\bar{Y}$ = $80. Now suppose he is offered a negative income tax plan that guarantees him $60 a week if he has no earned income, and "taxes" any earned income at 50 per cent (t=.5) by reducing the guarantee as earned income rises. This plan has a "breakeven" at point B (greater than $\bar{Y}$) corresponding to an earned income of $120 = $60/t; to the left of this point he receives no payments. The negative income tax plan has increased his opportunity set from AC to ABC. Under conventional conditions (in particular $\alpha_1 = \alpha_2$), he would move to a new equilibrium point such as D corresponding to <u>both</u> a higher income and more leisure.

Now suppose $\alpha_2 > \alpha_1$ which may realistically be the case if for example the individual is a student who needs $80 a week, but who is interested mainly in devoting more time to studying. Initially at point C he has reached a kink at which

$$\frac{\alpha_0}{\alpha_1}\left(\frac{80}{L}\right) = \lim_{Y \to 80^-}\left(-\frac{dY}{dL}\right) > w > \lim_{Y \to 80^+}\left(-\frac{dY}{dL}\right) = \frac{\alpha_0}{\alpha_2}\left(\frac{80}{L}\right)$$

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[3]There is precedent for using a Cobb-Douglas function in this type of problem. See Berner [7] for details.

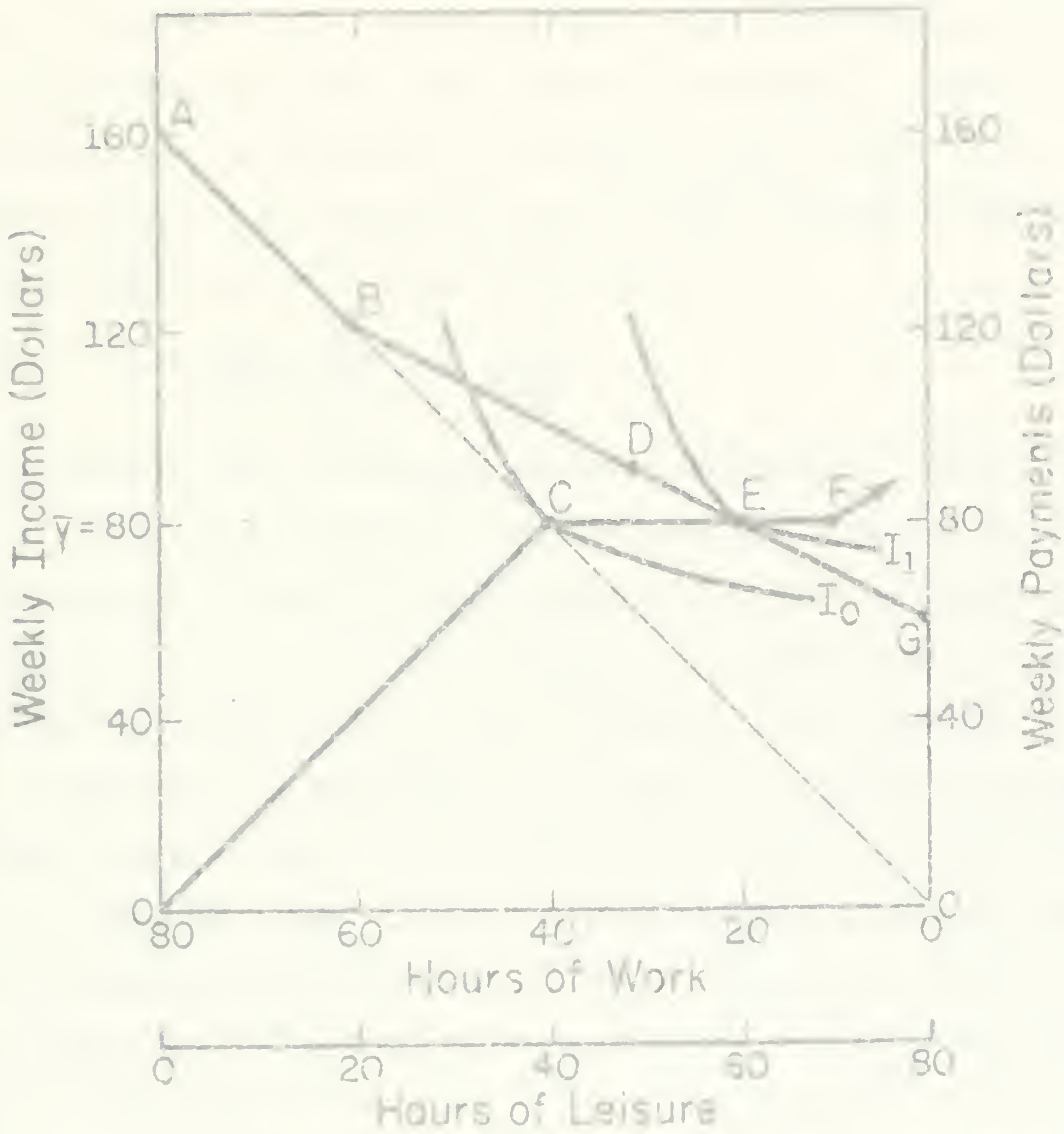

Figure 4: Response to a Negative Income Tax

where w = $2.00/hour is his original effective wage. The negative income tax plan changes his effective wage to $w(1-t)$ = $1.00/hour. As was seen in section 3, his "expansion path" will proceed along $\bar{Y}$ = $60 until he reaches a point F on an indifference curve for which the upper one-sided derivative $\left|\frac{dY}{dL^+}\right|$ equals his new effective wage, i.e.,

$$L^* = \frac{\bar{Y}}{w(1-t)} \left(\frac{\beta}{\alpha_2}\right) = 60 \left(\frac{\beta}{\alpha_2}\right).$$

He then proceeds along an expansion path with slope $(\alpha_2 w/\beta)$. However, if before the point F is reached he encounters his budget constraint ABG, he must then stop at a point E on indifference curve I_1 which corresponds to <u>no</u> change in income with the entire effect of the negative income tax plan manifested in increased leisure (i.e., a reduction in labor supply).

Intuitively, the explanation is as follows. The reduction of the effective wage as a result of the tax means that the price of leisure has, relatively speaking, dropped. Hence, he will consume more leisure. However, the marginal rate of substitution, MRS $= (-\frac{dY}{dL})$, for $Y \to (\bar{Y})^-$ is greater than the MRS for $Y \to (\bar{Y})^+$. Thus he will not "consume" any more earned income until it "pays" him to do so, i.e., until the MRS equals the price ratio (the effective wage). In this example the incentive effect of the negative income tax plan does not set in again until the point F is reached. How much leisure will be consumed in between depends critically on the change $\alpha_1 - \alpha_2$ in elasticities and the tax rate (provided he is below his break-even point).

5. Empirical Application

Following the suggestion of a referee, a CDS production function has been estimated based on the classical study of steam-generating electrical utilities by Nerlove [9].[a] One of the distinguishing features of Nerlove's study was his conclusion that the returns to scale appeared to be a decreasing function of the level of output. In the framework of a CDS production function, another equally possible interpretation would have been that the returns to scale were a decreasing function of the levels of inputs.

In light of recent world developments and the "energy crisis," the question of whether there exists varying returns to scale in the output of electricity takes on added importance. Increasing returns to scale over relevant ranges of output for public utilities have important implications in terms of public subsidies and investment policies in such an industry. In order to investigate the returns to scale, Nerlove suggested that public utility firms can be viewed as taking their output and input prices as exogeneous, and then minimizing their total cost of production subject to a Cobb-Douglas production function. The data used to estimate this model was based on 145 firms in 1955, and it can be found in Nerlove [9] and the sources cited therein.

Suppressing firm subscripts it is assumed here that firms have production functions of the form

(15) $Q = \delta_i L^{\alpha} K^{\beta} F^{\gamma} \epsilon_i$

[a]Partly due to Nerlove's [9, p. 175] own warnings concerning the accuracy of his data, and partly due the recent work of Greene [4], which questions whether the underlying production function is even homothetic, the results presented here should be viewed mainly as an illustrative example.

where Q is output (measured in billions of kwh), L is the labor input, K is the capital input, F is the fuel input (principally coal), ε is a residual expressing neutral variations in efficiency among firms, and β_1, α, η, and δ_1 are the conventional parameters of a Cobb-Douglas production function. Unlike Nerlove it will be assumed here that the output elasticity of fuel equals δ_1, if $F \leq \overline{F}_1$, or δ_2 if $F > \overline{F}_1$. The choice of the fuel knot, $\overline{F}_1 = 61718.4$ (millions of Btu's), is admittedly somewhat arbitrary. Hopefully, in more elaborate applications the choice of $\overline{F}_1$, or for that fact, labor and capital knots, will reflect technological considerations of the production process. The choice here is no more arbitrary than the five group breakdown used by Nerlove, and it corresponds to approximately the 33rd percentile of the firms' fuel input level.[5]

Following Nerlove [9 , pp. 171-175] it can be showed that cost minimization subject to (15) and exogenous output and input prices implies the cost function

$$(16) \quad \ln C - P_F = \lambda_1 + \frac{1}{w_1} \ln Q + \frac{\alpha}{w_1} (\ln P_L - \ln P_F) + \frac{\eta}{w_1} (\ln P_K - \ln P_F) - \frac{1}{w_1} \ln \varepsilon$$

where $w_1 = \alpha + \eta + \delta_1$ is the returns to scale, C is the total cost, P_L is the price of labor, P_K is the price of capital, P_F is the price of fuel, and

$$\lambda_1 = \ln w_1 - \frac{1}{w_1} \left[\ln \beta_1 + \alpha \ln \alpha + \eta \ln \eta + \delta_1 \ln \delta_1 \right].$$

Proceeding as in Nerlove's Model A which assumes that P_K is the same for all firms, (16) implies

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[5] While beyond the scope of this illustrative example, there appear to be ways to handle cases in which the knots are unknown parameters to be estimated. While the techniques developed so far have been for simpler models, they can at least in theory be applied to more complicated models. See Hudson [8], Hinkley [6], and Halpern [5] for details.

Table 1

Coefficient Estimates for Nerlove Example

Coefficient	Estimated Value	Estimated Standard Error
α	.5842*	.2533
δ_1	.5905**	.1240
δ_2	.5591**	.1244
η	.01574	.2394
θ	1823.	3498.

"*" denotes significance at the 5% level.

"**" denotes significance at the 1% level.

(17) $$w_i \ln C = \beta_i + \ln Q + \alpha \ln P_L + \gamma \ln P_F - \ln \varepsilon$$

where $\beta_i = w_i \lambda_i + \eta \ln P_K$. Note that in order for the disturbance term in (17) to be homoscedastic when returns to scale vary (i.e., $\delta_1 \neq \delta_2$), (16) was first multiplied through by w_i.

Under the assumption that the residuals in (17) are independent, homoscedastic, and normally distributed, the likelihood function corresponding to (17) can be maximized subject to the continuity constraint $\ln \theta_2 = \ln \theta_1 + (\delta_1 - \delta_2) \ln \overline{F}_1$. The resulting maximum likelihood estimators $\hat{\alpha}$, $\hat{\delta}_1$, $\hat{\delta}_2$, $\hat{\eta}$, and $\hat{\theta}_1$ will asymptotically be consistent, normally distributed, and efficient. Since the likelihood function is nonlinear in these parameters, it is necessary to employ a technique such as Marquardt's method of steepest descent to obtain the maximum.

The actual maximum likelihood estimators (together with their standard errors) are given in Table 1. The results are similiar to those of Nerlove: the output elasticities of labor and fuel highly significant, and the output elasticity of capital is insignificant (as it often was in Nerlove's study). Of course the result of principal concern is the estimated change $\hat{\delta}_1 - \hat{\delta}_2 = .03140$ in the output elasticity of fuel at $\overline{F}_1$. The asymptotically normal test statistic for testing the significance of this change is

$$\frac{\hat{\delta}_1 - \hat{\delta}_2}{\left[\text{Est. Var}(\hat{\delta}_1) + \text{Est. Var}(\hat{\delta}_2) - 2 \text{ Est. Cov}(\hat{\delta}_1, \hat{\delta}_2)\right]^{1/2}} = 3.112$$

which is highly significant. Thus, while there are increasing returns to scale everywhere, they appear to undergo a significant drop along $\overline{F}_1$.

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